in fishes but in Amphibia and reptiles, would suggest that the cause of the transformation of longitudinal stripes into spots on the lumbar and sacral regions of lizards is the result of the same specializing growth-force. It may perhaps be regarded as a surviving remnant of the segment-forming force, which has affected the pigment bands in a manner identical in the vertebrates and insects. This transformation of stripes into spots, and the fusion of two dorsal tubercles into a median one, may be, then, the sign of some latent or surviving amount of force concerned in the origin and formation of segments, which crops out in the larval stages of insects and in young lizards, resulting in this opisthenogenetic mode of origin of spots from bands.

ORTHIC CURVES; OR, ALGEBRAIC CURVES WHICH SATISFY LAPLACE'S EQUATION IN TWO DIMENSIONS.

BY CHARLES EDWARD BROOKS, A.B.

(*Read May 20, 1904.*)

I propose a study of the metrical properties of algebraic plane curves which are apolar, or, as it is sometimes called, harmonic, with the absolute conic at infinity. If we disregard the right line, the simplest orthic curve is the equilateral (conic) hyperbola, and the name equilateral hyperbola is sometimes extended to orthic curves of higher order. Doctor Holzmüller,[1] who devotes a section to curves of this kind, calls them hyperbolas; and M. Lucas[2] calls them "stelloides." M. Paul Serret, in a series of three papers in *Comptes Rendus*,[3] uses the word "équilatère" for a curve with

[1] *Einführung in die Theorie der Isogonalen Verwandschaften und der Conformen Abbildungen*, Gustav Holzmüller, Leipzig, 1882, p. 202. . . .

[2] "Géométrie des Polynomes," Felix Lucas, *Journal de l'Ecole Polytechnique*, 1879, t. XXVIII.

[3] *Comptes Rendus*, 1895, t. 121. Sur les hyperboles équilatères d'ordre quelconque, p. 340.

Sur les faisceaux regulieres et les équilatères d'ordre n. p. 372.

Sur les équilatères comprises dans les equations

$$0 = \Sigma_1^{2n-2} l_1 q_1^n = H_n,$$

$$0 = \Sigma_1^{2n-1} l_1 q_1^n = H_n + \lambda H'.$$

p. 438.

asymptotes concurrent and parallel to the sides of a regular polygon. It seems advisable to follow M. Serret's usage, and to denote such a curve by the name equilateral, using another term to express apolarity with the absolute. For this purpose I have adopted the word orthic.

If we use Cartesian coördinates, a curve

$$U(XY) = 0,$$

is apolar with the absolute conic,

$$\xi^2 + \eta^2 = 0,$$

if

$$\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} = 0.$$

In other words, an orthic curve is one which satisfies Laplace's equation in two dimensions.

Part One—The Orthic Cubic Curve.

I. *The Condition that a Curve be Orthic.*

In the analysis which may be required, I shall employ conjugate coördinates, x, $\bar{x}$, which may be defined as follows: If X and Y are rectangular Cartesian coördinates of any point, the conjugate coördinates of that point are

$$x = X + iY, \ \bar{x} = X - iY,$$

when the origin is retained, and the axis of X is chosen as the axis of reals, or base line. It is sometimes convenient to think of x as the vector from the origin to the point, and of $\bar{x}$ as the reflection of this vector in the base line. If x, $\bar{x}$ is a real point of the plane, not on the base line, $x - \bar{x} = 0$, x and $\bar{x}$ are conjugate complex numbers. Since if one of its coördinates is known the other is immediately obtainable, we shall, as a rule, name a point by giving only one of its coördinates. It is convenient to reserve the letters t and τ for points on the unit circle,

$$x\bar{x} = 1.$$

Now, Laplace's equation,

$$\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} = 0,$$

when applied to a function of x and $\bar{x}$, becomes

$$\frac{\partial^2 U(x\bar{x})}{\partial x \partial \bar{x}} = 0.$$

It follows that:

The necessary and sufficient condition that a curve be orthic is that its equation in conjugate coördinates contain no product-term.

II. *Kinematical Definition of the Orthic Curve.*

Let us now proceed to the study of the orthic curve of the third order. I shall obtain the equation of an orthic cubic in a way which will suggest immediately a method for the construction of points on the curve.

The path of a point which moves in such a way that it preserves a constant orientation from three fixed points is an orthic cubic curve.

If x is the moving point, and the three fixed points are α, β, γ, then the sum of the amplitudes of the strokes which connect x with α, β, γ, must remain constant. That is, we must have

$$(x - \alpha)(x - \beta)(x - \gamma) = \rho\tau_1.$$

If the curve is to be real, the conjugate relation,

$$(\bar{x} - \bar{\alpha})(\bar{x} - \bar{\beta})(\bar{x} - \bar{\gamma}) = \rho\tau_1^{-1},$$

must hold simultaneously.

The equation of the curve is obtained by eliminating the parameter ρ between these. It is

$$x^3 - (\alpha + \beta + \gamma)\, x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)\, x - \alpha\beta\gamma$$

$$= \tau_1^2 \{\bar{x}^3 - (\bar{\alpha} + \bar{\beta} + \bar{\gamma})\, \bar{x}^2 + (\overline{\alpha\beta} + \overline{\beta\gamma} + \overline{\gamma\alpha})\, \bar{x} - \overline{\alpha\beta\gamma}\}.$$

This is the most general equation of the third degree which we can have without introducing the product. As a consequence it represents a perfectly general orthic cubic.

If we transform to

$$x = \tfrac{1}{3}(\alpha + \beta + \gamma),$$

the centroid of $\alpha\beta\gamma$, as a new origin, and so choose the base line that τ_1^2 is real, the equation takes the form

$$x^3 + a_0 x + a_1 + \bar{a}_0 \bar{x} + \bar{x}^3 = 0.$$

The equation of any orthic cubic can be brought to this form. The three points, α, β and γ, are on the curve, and form what it is convenient to call a triad of the curve.

III. *The Orthic Curve is an Equilateral Curve.*

Consider the orthic cubic,

$$x^3 - s_1 x^2 + s_2 x - s_3 = \tau_1^2 (\bar{x}^3 - \bar{s}_1 \bar{x}^2 + \bar{s}_2 \bar{x} - \bar{s}_3),$$

where the s's are the elementary symmetrical functions of α, β The approximation at infinity,

$$(x - \tfrac{1}{3}s_1)^3 - \tau_1^2 (\bar{x} - \tfrac{1}{3}\bar{s}_1)^3 = 0,$$

makes both the square and the cube terms vanish, and therefore represents the asymptotes. The factors of this are:

$$x - \tfrac{1}{3}s_1 - \sqrt[3]{\tau_1^2}\,(\bar{x} - \tfrac{1}{3}\bar{s}_1) = 0,$$

$$x - \tfrac{1}{3}s_1 - \omega . \sqrt[3]{\tau_1^2}\,(\bar{x} - \tfrac{1}{3}\bar{s}_1) = 0,$$

$$x - \tfrac{1}{3}s_1 - \omega^2 . \sqrt[3]{\tau_1^2}\,(\bar{x} - \tfrac{1}{3}\bar{s}_1) = 0.$$

where $\omega^3 = 1$.

These three lines meet at the point

$$x = \tfrac{1}{3}(\alpha + \beta + \gamma)$$

which we may call the centre of the curve. We notice that:

The centre of the orthic cubic is the centroid of the triad.

The clinants of the asymptotes are $\tau_1^{\frac{2}{3}}$, $\omega\tau_1^{\frac{2}{3}}$, $\omega^2\tau_1^{\frac{2}{3}}$. They differ only by the constant factor ω. Now we know that multiplying the clinant of a line by ω is equivalent to turning the line through an angle $\frac{2\pi}{3}$. A rotation $\frac{2\pi}{3}$ about the centre sends each asymptote into another. It follows that the asymptotes of an orthic cubic are concurrent and parallel to the sides of a regular triangle. M. Serret[1] calls such a figure of equally inclined lines which meet in a point a regular pencil, and a curve with asymptotes forming a regular pencil he calls an "*équilatère.*"

[1] *Comptes Rendus*, Sur les hyperboles équilatères d'ordre quelconque. 1895, t. 121, p. 340.

Now any cubic curve, the asymptotes of which form a regular pencil, can be brought to the form :

$$x^3 + a_0 x + a_1 + \overline{a_0 x} + \bar{x}^3 = 0,$$

in which we recognize it as orthic. It follows that:

The orthic cubic and the equilateral of order three are identical.

The relation

$$(x - \alpha)(x - \beta)(x - \gamma) = \rho\tau_1 = z$$

may be regarded as mapping a line through the origin in the z plane,

$$z - \tau_1^2 \bar{z} = 0$$

into the orthic cubic. We are thus able to identify the latter with the curves discussed by Holzmüller[1] and by Lucas.[2]

IV. *Construction of Points of an Orthic Cubic.*

A figure of the orthic cubic may be obtained without great difficulty by constructing points of the curve. In order to show how this may be done, it is necessary to prove the following lemma:

Elements of the pencil of equilateral (orthic) hyperbolas, of which the stroke $\beta\gamma$ is a diameter, intersect corresponding elements of the pencil of lines through α on an orthic cubic of which $\alpha\beta\gamma$ is a triad.

For the line through α,

$$(x - \alpha) = \rho\tau',$$

and the equilateral hyperbola on $\beta\gamma$ as a diameter,

$$(x - \beta)(x - \gamma) = \rho\tau'',$$

intersect on the orthic cubic

$$(x - \alpha)(x - \beta)(x - \gamma) = \rho\tau_1$$

[1] Holzmüller, *Conformen Abbildungen*, p. 205.

[2] Lucas, Géométrie des Polynomes, *Journal de l'Ecole Polytechnique*, t. XXVIII, p. 23.

if

$$\tau'\tau'' = \tau_1.$$

If the two pencils are given, it is only necessary to pair off lines and curves according to the relation

$$\tau'\tau'' = \tau_1,$$

and to mark intersections. These will be points of the curve.

A very simple instrument for drawing the equilateral hyperbolas required in the construction is made in the following way: Two toothed wheels of equal diameters are attached beneath the drawing

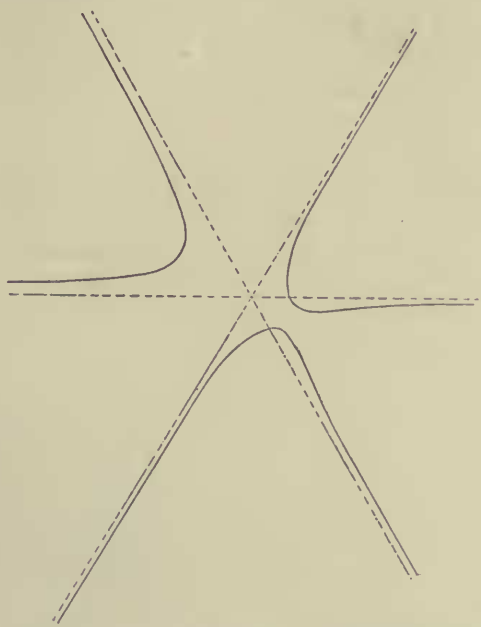

Figure 1. A unipartite orthic cubic which has three real inflections, one of which is at infinity.

board in such a way that their teeth engage. The axles are perpendicular to the board and come through it at β and γ. The axles, which turn with the wheels, carry long hands or pointers which sweep over the board. On account of the cogs, the wheels can turn only through equal and opposite angles. As a consequence, x, the intersection of the hands, has a constant orientation from β and γ, and in fact generates the orthic curve of the second order given by

$$(x-\beta)(x-\gamma)=\rho\tau',$$

which is the hyperbola required.

V. *Mechanical Generation of an Orthic Cubic.*

A mechanism which will actually draw an orthic cubic is very much to be desired. One might be made in some such way as the following: Suppose three hands like those described above (IV) to be pivoted at α, β and γ. Let them be held together in such a way that, while each is free to move along the others, they must always meet in a point, which is to be the tracing point. Each hand is to receive its motion from a cord wound about a bobbin on its axle. The bobbins are to be equal in diameter. The cords pass through conveniently placed pulleys, and are kept tight and vertical by small equal weights at their ends. Consider, to fix ideas, those three weights which by their descent give the hands positive rotation. If, now, the tracing point be moved along an orthic cubic which has α, β, γ for a triad, the total turning of the bobbins will be zero, and as a consequence the total descent of the weights will be zero. Conversely, if we can move these vertically and in such a way that the total descent will be zero, the tracing point can move only along an orthic cubic. This result will be obtained if the centre of gravity of the three weights can be kept fixed. It will not do, however, to connect the three weights by a rigid triangle pivoted at its centre of gravity, for then they will not move vertically. But since a parallel projection does not alter the centroid of a set of points, the desired result will be attained if the weights are constrained to vertical motion by guides of some kind, and are kept in a plane which always passes through the centre of gravity of one position of the weights.

VI. *The Orthic Cubic through Six Points of a Circle.*

Consider the general orthic cubic given by

$$x^3-a_0x^2+a_1x-a_2+a_3\bar{x}-a_4\bar{x}^2+a_5\bar{x}^3=0.$$

It cuts the unit circle,

$$x\bar{x}=1,$$

in six points, the roots of

$$x^6 - a_0x^5 + a_1x^4 - a_2x^3 + a_3x^2 - a_4x + a_5 = 0.$$

If we want the cubic to meet the circle in six given points, say τ_1, $\tau_2, \ldots \tau_6$, then this equation must be identical with

$$x^6 - s_1x^5 + s_2x^4 - s_3x^3 + s_4x^2 - s_5x + s_6 = 0,$$

in which the s's stand for the elementary symmetrical combinations of the six τ's. This requires

$$a_0 = s_1,\ a_1 = s_2,\ a_2 = s_3,$$

$$a_3 = s_4,\ a_4 = s_5,\ a_5 = s_6$$

The coefficients of the cubic equation are then precisely determined, with the result that:

But one orthic cubic can be constructed through any six points of a circle.

It remains for us to show that one such curve can always be drawn: that is, that the equation

$$x^3 - s_1x^2 + s_2x - s_3 + s_4\bar{x} - s_5\bar{x}^2 + s_6\bar{x}^3 = 0$$

always represents a real curve. If we so choose the base line that $s_6 = 1$ then we have

$$\bar{s}_1 = s_{6-1}\, s_6^{-1} = s_{6-1},$$

and the equation takes the form

$$x^3 - s_1x^2 + s_2x - s_3 + \overline{s_2x} - \overline{s_1x^2} + \bar{x}^3 = 0,$$

which is, obviously, self-conjugate, and is therefore satisfied by the coördinates of real points. As a result:

An orthic cubic can always be drawn through six points of a circle. It is then determined uniquely.

VII. *The Intersections of an Orthic Cubic with a Circle.*

When the orthic cubic is referred to the six points in which it cuts the unit circle, the equations of the asymptotes take the form

$$x - \tfrac{1}{3}\, s_1 = (-s_6)^{1/3}\omega^i\, (\bar{x} - \tfrac{1}{3}\, s_5s_6^{-1}).$$

$$i = 0,\ 1,\ 2.$$

These three lines meet at

$$x = \tfrac{1}{3} s_1,$$

the centre. This point, the origin, and the point which is the centroid of the six points on the circle lie on a line; and the latter point is midway between the other two. This leads to the interesting fact that:

The centroid of the six points in which any circle meets an orthic cubic bisects the stroke from the centre of the curve to the centre of that circle.

VIII. *Triads of the Curve.*

We spoke of the three points α, β, γ, which have the same orientation from every point of the curve, as a triad of the curve. Let us see how many such triads there are, and how they are arranged. The relation

$$(x - \alpha)(x - \beta)(x - \gamma) = z$$

may be regarded as establishing a correspondence between points x in one plane and points z in another plane, in such a way that if z describe a line ξ through the origin, the point x generates an orthic cubic on $\alpha\beta\gamma$ as a triad. To every position of z on the director line ξ there correspond three points in the x-plane. I shall show that each such set of three points is a triad. Write

$$F(x) = (x - \alpha)(x - \beta)(x - \gamma).$$

Then, if x_1, x_2, x_3, are the three points which correspond to z,

$$F(x) - z = (x - x_1)(x - x_2)(x - x_3).$$

And also

$$F(x) - z' = (x - x_1')(x - x_2')(x - x_3').$$

Now this relation is satisfied by x_1, or x_2, or x_3.

$$F(x_1) - z' = (x_1 - x_1')(x_1 - x_2')(x_1 - x_3') = z - z'.$$

Since $z - z'$ is a point of the director line, it follows that the three points x_1', x_2', x_3', which correspond to any point z' of the director line, have the same orientation from every point of the curve. We conclue that:

To every point of the director line corresponds a triad; all the *points of the curve have the same orientation from any triad, and all the triads of the curve have the same orientation from any point of the curve.*

IX. *The System of Confocal Ellipses Connected with the Triads.*

We seek the points of a triad which correspond to a given point z.[1] The map equation can be brought to the form

$$x^3 - 3x = 2z$$

by choosing the centre of the curve as a new origin and making a suitable choice of the unit stroke. We see at once that the sum of the x's for a given z is zero. In other words: *The centroid of any triad is the centre of the cubic.*

Making use of the method known as Cardan's solution, put

$$x = \mu t + v,$$

where μ is real. Then

$$x^3 - 3x = 2z$$

becomes

$$\mu^3 t^3 + v^3 + 3\mu^2 t^2 v + 3\mu t v^2 - 3\,(\mu t + v) = 2z$$

And we have as two relations between v and μt,

$$2z = \mu^3 t^3 + v^3,$$

and

$$(\mu t + v)\;(\mu t v - 1) = 0.$$

When z is zero, the values of x are $\pm\sqrt{3}$ and 0; and when z is not zero, we must have

$$v = \frac{1}{\mu t}.$$

This leads to the expression of x and z in terms of μt as follows:

$$x = \mu t + \frac{1}{\mu t},$$

$$2z = \mu^3 t^3 + \frac{1}{\mu^3 t^3}.$$

Now if we assign any value to μ, and let t run around the unit circle, x describes an ellipse with foci at $x = +2$ and $x = -2$.

[1] Harkness and Morley, *A Treatise on the Theory of Functions*, p. 39.

But at the same time, z also describes an ellipse with its foci at $z = +1$ and $z = -1$. These two ellipses are related in such a way that a point z on one of them is correlated by the equation

$$x^3 - 3x = 2z$$

with three points on the other. Now the foci of both these ellipses are independent of the particular value of μ selected; it follows that if we assign successive values to μ, we shall obtain in each plane a system of confocal ellipses of such a sort that the equation

$$x^3 - 3x = 2z$$

establishes a one to one correspondence between them. In each plane the origin is the centre of all the ellipses. Applying this scheme to the case in hand, we see that a triad must be inscribed in one of the ellipses in the x-plane. But the centroid of the triad is the centre of the ellipse; so the ellipse must be the circumscribed ellipse of least area of that triad. We may say, then, that:

The triads of the orthic cubic are cut out on the curve by a particular system of confocal ellipses, and each ellipse is the circumscribed ellipse of least area of the triad on it.

X. *The Riemann Surface for an Orthic Cubic.*

If we examine the equation

$$x^3 - 3x = 2z$$

for equal roots, we find that the double points of the x-plane are at $x = +1$ and at $x = -1$. These values of x correspond to the branch points in the z-plane, $z = +1$ and $z = -1$.

Let us for a moment replace the z-plane by a three-sheeted Riemann surface. All three sheets must hang together at infinity, and two sheets at each of the branch points. Let the first and second sheets be connected by a bridge along the base line from $+1$ to infinity, and the second and third sheets be similarly connected by a bridge along the real axis from -1 to infinity.

Select on this surface any large ellipse with foci at the branch points, and any line as a director line. Now consider the contour obtained by starting from a point of this inside the ellipse, going thence along the line to meet the ellipse, along an arc of the ellipse to meet the line, and then along the line to the point of departure.

We can choose this path in such a way that one of the following three cases must arise:

(1) *The contour passes through a branch point.*

(2) *The contour surrounds two branch points.*

(3) *The contour surrounds no branch point.*

In case (1) we know that the cubic must have a node. In the second case, by going three times around we can pass continuously through every sheet of the Riemann surface and therefore through every value of x. Or, thinking again of the x-plane, we have a unicursal boundary. Now it happens that the ellipse we choose maps into one and not three ellipses on the x-plane. If we imagine this to expand indefinitely we shall have to consider the boundary as our orthic cubic. It follows at once that:

The orthic cubic which corresponds to a line which does not pass between the branch points is unipartite.

If the contour includes one branch point, and therefore crosses one bridge of the Riemann surface, we must go along two unconnected curves to reach all the values of x. When these two curves are spread on the x-plane they lead at once to the conclusion that:

The orthic cubic which corresponds to a line which passes between the branch points is a bipartite curve.

XI. *Triads in Special Cases.*

Let us turn our attention again to the two planes connected by the relation

$$x^3 - 3x = 2z$$

We notice that while the ellipses in the z-plane have their foci at the branch points, the foci of the corresponding system of ellipses are not the double points of the x-plane, but are the points $x = +2$ and $x = -2$, each of which, with one of the double points counted twice, forms a triad.

As a rule there are two triads of the curve on each ellipse, corresponding to the two points in which the director line cuts an ellipse of the system in the z-plane. But unless the line go between the branch points it will be tangent to one ellipse, consequently two triads will coincide, and the cubic will be tangent at three places to one of the ellipses of the system. No part of the cubic can be inside of that ellipse.

When μ is 1, the two ellipses degenerate into two segments,

$$x = t + t^{-1} \text{ or } \overline{2, -2},$$

$$2z = t^3 + t^{-3} \text{ or } \overline{1, -1}.$$

If the line pass between the branch points, and so cut the segment $\overline{1, -1}$, two triads again coincide, but in this case the three points lie on a line, and we do not have the triply tangent ellipse.

When the line ξ cuts the axis of imaginaries,

$$z + \bar{z} = 0,$$

we have

$$z = \rho\, \varepsilon^{\frac{\pi i}{2}},$$

and

$$t^3 = \rho'\, \varepsilon^{\frac{\pi i}{2}}.$$

It follows that $am\ t = \frac{\pi}{6}$, and so ωt is the reflection of t in the axis of imaginaries and $\omega^2 t$ is a pure imaginary. Then, since we know that

$$x_i = \omega^i \mu t + \frac{1}{\omega^i \mu t},$$

$$i = 1, 2, 3,$$

we see that x_1 is the reflection of x_3 in the line $x + \bar{x} = 0$, and that x_2 is on that line. It follows that the triangle $x_1 x_2 x_3$ is isosceles and that its base $x_1 x_3$ is parallel to the real axis. There is again an isosceles triangle when t^3 is real. This triangle has its vertex on the axis of reals and its base perpendicular to that axis. From the discriminant of the quadratic in $\mu^3 t^3$,

$$z^2 - 4,$$

we see that t^3 is real when $z > \pm 1$. In other words, if the director line ξ cut the axis of reals, but not between the branch points, we have such an isosceles triangle.

From the above considerations, we see that if the director line is either of the axes

$$x + \bar{x} = 0,\ x - \bar{x} = 0,$$

then one branch of the orthic cubic must be a right line; the remaining portion of the curve must then be an ordinary hyperbola, and the inclination of its asymptotes must be either $\frac{\pi}{3}$ or $\frac{2}{3}\pi$. The first value refers to the case when the director line is the axis of imaginaries; and the last, to the case when it is the axis of reals.

XII. *The Intersections of the Circumscribed Circle of a Triad with the Cubic.*

Suppose we put a circle through the points of a triad, and ask, Where are the remaining three points in which it cuts the cubic? For convenience, let three points of the unit circle be taken as a triad. The cubic is then

$$(x - t_1)(x - t_2)(x - t_3) = \tau_1^2 (\bar{x} - t_1^{-1})(\bar{x} - t_2^{-1})(\bar{x} - t_3^{-1}).$$

On eliminating $\bar{x}$ from this and the equation of the circle we obtain

$$(x - t_1)(x - t_2)(x - t_3) = \frac{\tau_1^2 (t_1 - x)(t_2 - x)(t_3 - x)}{t_1 t_2 t_3 x^3},$$

or

$$x^3 = \frac{-\tau_1^2}{t_1 t_2 t_3},$$

as the equation of the three points sought. The roots of this,

$$x_1 = x,\ x_2 = \omega x,\ x_3 = \omega^2 x,$$

are the coördinates of the vertices of an equilateral triangle. As there is no restriction in taking the triad on the unit circle, we have the following theorem:

If a circle cut an orthic cubic in a triad, then the two curves have three other intersections, which form an equilateral triangle.

XIII. *The Pencil of Orthic Cubics Which Have a Triad in Common.*

We have seen that the relation

$$(x - \alpha)(x - \beta)(x - \gamma) = z$$

maps a line through the origin into an orthic cubic of which $\alpha\beta\gamma$ is a triad. It must then map all the lines through the origin into a single infinity of orthic curves[1] which have the common triad $\alpha\beta\gamma$.

[1] Felix Lucas, *Journal de l'Ecole Polytechnique*, t. XVIII, p. 21.

If we regard τ as a parameter, we may say that

$$(x-\alpha)(x-\beta)(x-\gamma)=\tau(\bar{x}-\bar{\alpha})(\bar{x}-\bar{\beta})(\bar{x}-\gamma)$$

is the equation of the pencil of orthic cubics which have the triad $\alpha\beta\gamma$. It will be convenient to give a pencil of this sort some name; let us refer to it as a *central pencil*, noting for our justification that the centroid of the triad is the centre of every curve of the pencil.

If there were any real point other than α, β, or γ, on two curves of this pencil, it would map into a real point of the z-plane, not the origin, which would be on two of the lines through the origin. As this is manifestly impossible, it follows that:

Two orthic cubics which have a triad in common have no other real intersection.

Now we know that two cubic curves intersect in nine points, and that if the curves given by the equation

$$(x-\alpha)(x-\beta)(x-\gamma)=\tau(\bar{x}-\bar{\alpha})(\bar{x}-\bar{\beta})(\bar{x}-\bar{\gamma})$$

really constitute a pencil, there must be six imaginary points whose coördinates satisfy this equation whatever the value of τ. Let us form the following table of coördinates. The real intersections are

$$x_1=\alpha,\ \bar{x}_1=\bar{\alpha},$$

$$x_2=\beta,\ \bar{x}_2=\bar{\beta},$$

$$x_3=\gamma,\ \bar{x}_3=\bar{\gamma}.$$

It is evident that each of the following points:

$$x_4=\alpha,\ \bar{x}_4=\bar{\beta},$$

$$x_5=\alpha,\ \bar{x}_5=\bar{\gamma},$$

$$x_6=\beta,\ \bar{x}_6=\bar{\alpha},$$

$$x_7=\beta,\ \bar{x}_7=\bar{\gamma},$$

$$x_8=\gamma,\ \bar{x}_8=\bar{\alpha},$$

$$x_9=\gamma,\ \bar{x}_9=\bar{\beta},$$

satisfies the equation, independently of τ. These points, the six imaginary intersections of the pencil, are the antipoints[1] obtained

[1] Cayley, *Collected Mathematical Papers*, Volume VI, p. 499.

by selecting pairs in all possible ways from α, β, γ.

The figure of nine points in which two orthic cubics intersect may be regarded as an extension of the orthocentric four-point determined by two equilateral hyperbolas. It is convenient to extend the term *orthocentric* to such a figure. Resuming the results obtained above, we have:

When three of the points of an orthocentric nine-point are a triad of the orthic curves through the nine points, the remaining six points are imaginary, and are the antipoints of the three real points. The centroid of the nine points is the centre of every orthic cubic through them.

It is convenient to speak of a set of orthocentric points determined by a central pencil as a *central* set. Since any three points determine a pencil of orthic cubics of which they are a triad, any three points, with all their antipoints, form a central orthocentric nine-point.

XIV. *The Foci.*

We shall now attack the problem of finding the foci of the orthic cubic. Let us begin with a few words as to the way in which the foci of a curve appear in analysis with conjugate coördinates. The focus of a curve is the intersection of a tangent from one circular point with a tangent from the other circular point. In other words, if the circular rays from a point are tangent to a curve, that point is a focus of the curve. Now the equation of the circular rays from a point a, $\bar{a}$, is

$$(x - a)(\bar{x} - \bar{a}) = 0.$$

Therefore, one of the lines is

$$x - a = 0,$$

and the other is

$$\bar{x} - \bar{a} = 0.$$

Suppose the equation of the curve is

$$F(x\bar{x}) = 0.$$

Now if the circular ray

$$x - a = 0$$

is tangent to the curve, then

$$F(a\bar{x}) = 0,$$

the eliminant of x between these two, will have equal roots. But since the equation of a real curve must be self-conjugate, if this has two coincident roots, then

$$F(\bar{a}x) = 0$$

must also have, and the point a, $\bar{a}$, is a focus. It follows that to find the foci of a curve, we have merely to find those values of x which make two values of $\bar{x}$ coincide. They are the vectors of the foci. Let us apply this method to the orthic cubic. The equation may be taken in the form

$$x^3 - 3x = 2z = a_0 + \lambda a_1$$

where λ is a real parameter and the director line is

$$a_0 + \lambda a_1 = 2z, \ \bar{a}_0 + \lambda \bar{a}_1 = 2\bar{z}.$$

These relations imply the conjugate expression

$$\bar{x}^3 - 3\bar{x} = 2\bar{z} = \bar{a}_0 + \lambda \bar{a}_1.$$

Two values of $\bar{x}$ become equal when $D_{\bar{x}}\bar{z} = 0$, *i.e.*, when

$$\bar{x}^2 - 1 = 0,$$

or

$$\bar{x} = \pm 1.$$

These values of $\bar{x}$ occur when

$$\bar{a}_0 + \lambda \bar{a}_1 = \pm 2,$$

or

$$\lambda = \frac{-\bar{a}_0 \pm 2}{\bar{a}_1}.$$

Either of these values of λ when substituted in

$$x^3 - 3x = a_0 + \lambda a_1$$

gives three points which are foci of the cubic.

There are, in general, six real foci, which fall into two sets of three. Each set of three corresponds to a single point of the z-plane, and is, therefore, a maximum inscribed triangle of one of the ellipses described above.

XV. *The Foci and the Branch Points.*

If we eliminate the parameter between

$$2z = a_0 + \lambda a_1$$

and

$$2\bar{z} = \bar{a}_0 + \lambda \bar{a}_1,$$

we get the equation of the line ξ,

$$\bar{a}_1 z - a_1 \bar{z} = a_0 \bar{a}_1 - a_1 \bar{a}_0.$$

Now suppose, for a moment, that this line does not contain either of the branch points $z = \pm 1$. Then, if we put $\bar{z} = \pm 1$ in the equation of the line and solve for z, we get a value which is not the conjugate of $\bar{z}$, but is the vector of the reflection of the point $z = \pm 1$ in the line considered. The three points in the x-plane got by putting

$$\lambda = \frac{-\bar{a}_0 \pm 2}{\bar{a}_1}$$

in the equation

$$x^3 - 3x = 2z$$

are the points mapped in the z-plane by the reflection of $z = \pm 1$ in the line ξ. It follows that:

The real foci of the orthic cubic which corresponds to a given line are the six points which correspond to the reflections in that line of the branch points.

If the director line pass through one of the branch points (*i.e.*, if $\frac{-\bar{a}_0 \pm 2}{\bar{a}_1}$ is real), two foci coincide to form the node, and the remaining one of the set is on the curve. One who looks at the matter from the point of view of the Riemann surface might be surprised that a branch point is to be reflected in the line in each sheet of the surface and not in the two sheets alone which it connects. A moment's consideration will show that whether or not two $\bar{x}$'s coincide depends on λ alone, and that either of three values of x give λ a particular value. It is clear that the reflection must be in every sheet of the surface.

In general, the orthic cubic is of class six. Since it cuts the line at infinity in three points apolar with the circular points, it cannot contain one of the circular points unless it is as a point of inflection. There should, therefore, be six tangents from each of the circular points and, consequently, thirty-six foci. The thirty foci still to be accounted for are the antipoints[1] of the six real foci, paired in all possible ways. When the cubic has a node it is of class four, and has but four real foci. The node takes the place of the two foci which coincide there.

The circular rays

$$x - a_1 = 0$$

and

$$\bar{x} - \bar{a}_2 = 0$$

meet at a_1, $\bar{a}_2$. So the thirty-six foci of an orthic cubic may be represented by the scheme of coördinates:

$$a_i, \ \bar{a}_j,$$

where i and j run from one to six. It follows that the centroid of the whole thirty-six is the centroid of the six real points; that is, the centre of the cubic.

Consider any selection of three foci. All their antipoints are foci, and the nine points together make up a central orthocentric set.

XVI. *The Foci of the Orthic Cubics which Have a Triad in Common Lie on Two Cassinoids.*

The foci of all the orthic cubics which have a common triad $\alpha\beta\gamma$ lie on two cassinoids which have their foci at α, β, and γ, and are orthogonal to the orthic curves.

We know that these cubics correspond to all the lines through a point, and that their foci correspond to the reflections of the branch points in those lines. Now the reflections of a fixed point in all the lines through a second point lie on a circle which goes through the first point, and which has its centre at the second point. Accord-

[1] Salmon, *Higher Plane Curves*, third edition, p. 122.

ingly, the foci of the cubics will lie on the curves which are the maps in the x-plane of two concentric circles in the z-plane. The centre of these circles maps into the triad common to all the cubics, and the circles themselves map into two cassinoids of the sixth order, about the triad, as M. Lucas[1] has shown. Each of the circles goes through one of the branch points, and, therefore, each of the cassinoids must have a node. If the point which corresponds to the triad $\alpha\beta\gamma$ is equidistant from the branch points, the two circles and also the two cassinoids coincide. In this case the cassinoid has two double points.

The lines which correspond to the cubics are all perpendicular to the circles which correspond to the cassinoids, and so, by the principle of orthogonality, the ovals are orthogonal trajectories of the cubics of the pencil.

XVII. *The Position of the Orthic Cubic in Projective Geometry.*

I shall close this study of the metrical properties of the orthic curve of the third order by showing that from the point of view of projective geometry the orthic cubic is really a general cubic. Any proper plane curve of the third order can be projected into an orthic curve.

We know that the points of contact of three of the six tangents to a cubic curve from any point of its Hessian lie in a line. Now these three points, considered as a binary cubic, have a Hessian pair. If this pair of points be projected to the circular points at infinity, the three tangents become equally inclined asymptotes, and continue to meet in a point. The cubic curve is then orthic and the transformation is accomplished. This projection requires two points to go into given points, and can, therefore, always be made. *In projective geometry the orthic cubic is any proper plane cubic.*

As an illustration of the way in which information about the orthic cubic applies to cubic curves in general, let us see what the characteristic property that the asymptotes are concurrent and equally inclined means. The circular points I and J are a pair of points apolar with the curve. Their join, the line at infinity, meets the curve in three points such that the tangents at these points meet

[1] Felix Lucas, Géométrie des Polynomes, *Journal de l'Ecole Polytechnique*, XXVIII, p. 5.

in a point, C, of the Hessian. Now we know[1] that such a line meets the Hessian in the point which corresponds to C. This leads to the theorems that:

The line joining two points apolar with a cubic curve meets the cubic in three points, the tangents at which meet in a point of the Hessian, and are apolar with the two points apolar with the curve.[2]

The line joining two points apolar with a cubic curve, and a tangent to the cubic at a point of this line, meet the Hessian of the given cubic in corresponding points.

A more novel result is the following. We have seen (XIV, p. 28) that the foci of an orthic cubic fall into two sets of three, in such a way that the two sets are triangles of maximum area inscribed in two confocal ellipses. Now if we consider tangents from I and J instead of foci, we have the following theorem:

If a *and* b *are a pair of points apolar with a cubic curve, then the tangents from either of these points, say* a, *fall into two sets of three in such a way that the line* ab *has the same polar pair of lines as to each set of three.*

PART TWO—ORTHIC CURVES OF ANY ORDER.

I. *Introduction.*

In the preceding pages we have studied the metrical properties of the orthic cubic in some detail. In the following portion of the work I shall indicate an extension of the more important results obtained in the study of the cubic to orthic curves of any order.

The general equation of the n^{th} degree between x and $\bar{x}$ contains $\frac{1}{2}n(n-1)$ product terms. If it is to represent an orthic curve the coefficients of these terms must be made zero. In other words, to make a curve of the n^{th} order orthic is equivalent to making it satisfy $\frac{1}{2}n(n-1)$ linear conditions. After this has been done there remain $2n$ degrees of freedom.

II. *The Orthic Curve is Equilateral.*

The kinematical definition which we obtained for the orthic cubic may be extended to curves of any order, that is:

[1] Salmon, *Higher Plane Curves*, third edition, articles 70 and 175.

[2] "On the Algebraic Potential Curves," Dr. Edward Kasner, *Bulletins of the American Mathematical Society*, June, 1901, p. 393.

The path of a point which moves so that its orientation from n *fixed points is constant is an orthic curve of order* n.

If $a_1, a_2, \ldots a_n$ are the fixed points, the condition on x is expressed by the relations

$$(x - a_1)\,(x - a_2)\,\ldots\,(x - a_n) = \rho\tau_1$$

and

$$(\bar{x} - \bar{a}_1)\,(\bar{x} - \bar{a}_2)\,\ldots\,(x - a_n) = \rho\tau_1^{-1}.$$

These lead to the equation of the curve,

$$x^n - s_1x^{n-1} + s_2x^{n-2}\,\ldots + \tau_1^2\,(\bar{s}_n\,\ldots + \bar{s}_1\bar{x}^{n-1} - \bar{x}^n) = 0,$$

where the s's are the elementary symmetric combinations of the a's. This is the general equation of an orthic curve. If we take $x = {}^1\!/_n s_1$ for a new origin, and make τ_1^2 real, the equation becomes

$$x^n + a_1x^{n-2} - a_2x^{n-3}\,\ldots - \overline{a_2x}^{n-3} + \overline{a_1x}^{n-2} + \bar{x}^n = 0.$$

The asymptotes are the n equally inclined lines given by the factors of the highest terms,

$$x^n + \bar{x}^n = 0.$$

These lines all pass through the origin; it follows that the centroid of the n points $a_1, \ldots a_n$, is the centre of the curve. Since every orthic curve can be brought to the above form, we see that every orthic curve is equilateral. The converse proposition, every equilateral is orthic, is not true. The general equation of an equilateral may be put in the form

$$x^n + a\bar{x}^n + \Phi\,(x\bar{x}) = 0,$$

where $\Phi\,(x\bar{x})$ is a perfectly general function of degree $n-2$. Φ contains $\frac{1}{2}(n-2)\,(n-3)$ product terms, which must vanish for the curve to be orthic. To make an equilateral curve orthic is, therefore, equivalent to making it satisfy $\frac{1}{2}(n-2)\,(n-3)$ linear conditions. For $n = 2$ and $n = 3$ this number is zero, so the equilateral conic and cubic are orthic. For the quartic, this says that to be orthic is one condition.

III. *N-ads, Foci, Intersections with a Circle.*

The relation

$$(x - a_1)\,(x - a_2)\,\ldots\,(x - a_n) = \rho\tau_1 = z$$

may be regarded as mapping a line through the origin in the z-plane into the orthic curve in the x-plane. The methods of analysis which were used, in the paragraphs referred to, in the study of the orthic cubic may be extended to any n, and lead to the following general theorems:

On an orthic curve of order n *there is a single infinity of sets of* n *points, n-ads of the curve, from which all points of the curve have the same orientation. All the n-ads have the same orientation from any point of the curve* (Part One, VIII).

Any n points may be taken as an *n-ad* of an orthic curve. If we take n points of the unit circle as an *n-ad*, and find the remaining intersections of the circle and the curve, we see that they are the vertices of a regular polygon (Part One, XII).

Every circle through an n-ad of an orthic curve of order n *meets the curve again in the* n *vertices of a regular polygon.*

The centre of an orthic curve is the centroid of every n-ad of the curve.

For when the equation is taken in the form

$$x^n + nx^{n-2} + \ . \ . \ . + a_{n-2}x = z$$

the origin is the centre of the curve, and is also the centroid of the n points which correspond to a point z. This equation will have two coincident roots whenever

$$D_x z \equiv nx^{n-1} + n(n-2)\, a_1 x^{n-3} \ . \ . \ . = 0.$$

In general, this will give $n - 1$ branch points in the z-plane. Each branch point, when reflected in the director line, gives rise to n real foci. If the line ξ revolve about a point, each reflection generates a circle (Part One, XIV). All $n - 1$ of these circles are concentric; and they map into $n - 1$ cassinoids, on which lie the foci of the curves which have the *n-ad* which corresponds to the centre of the system of circles. These cassinoids are orthogonal trajectories of the central pencil of orthic curves. Since each of the circles must contain a branch point, each cassinoid must have at least one node.

IV. *The Orthic Curve Referred to its Intersections with a Circle.*

We know that we may put $2n$ linear conditions on an orthic curve. If we make it go through $2n$ points on the unit circle, its equation, expressed in terms of the elementary symmetrical functions of the points where it meets the circle, becomes

$$x^n - s_1 x^{n-1} + s_2 x^{n-2} - + - s_{2n-1}\bar{x}^{n-1} + s_{2n}\bar{x}^n = 0.$$

The centre, found by equating to zero the $n-1^{st}$ derivative as to x, is

$$x = \tfrac{1}{n} s_1.$$

This is the midpoint of the stroke from the centre of the circle to the centroid of the $2n$ points. The equation of an asymptote now takes the form

$$x - \tfrac{1}{n} s_1 = \sqrt[n]{- s_{2n}}\,(\bar{x} - s_{2n-1}\, s_{2n}{}^{-1}).$$

V. *Construction of an Orthic Curve.*

The method which I have proposed (Part One, V) for the construction of an orthic cubic might be extended to the construction of any orthic curve. For this purpose the instrument must have n hands, moved by n weights. The centre of gravity of any number of weights could be held by joining them together in sets of three or less, and then joining again the centres of gravity of these sets. This operation could be repeated until the required number of weights is reached.

VI. *Geometrical Characteristics.*

The geometrical characteristics of an orthic curve of order n *are that it is equilateral, and that it intersects its asymptotes in points of a second orthic curve of order* n — 2.

For consider the orthic curve referred to its centre,

$$x^n + a_1 x^{n-2} - a_2 x^{n-3} \ . \ . \ . - \bar{a}_2\bar{x}^{n-3} + \bar{a}_1\bar{x}^{n-3} + \bar{x}^n = 0.$$

The asymptotes, which are given by

$$x^n + \bar{x}^n = 0,$$

are concurrent and equally inclined, so the curve is equilateral. The points common to the curve and its asymptotes lie on the curve

$$a_1 x^{n-2} - a_2 x^{n-3} + \ . \ . \ . - \bar{a}_2\bar{x}^{n-3} + \bar{a}_1\bar{x}^{n-2} = 0.$$

But this curve is of order $n-2$, and is orthic.

To require a curve to be equilateral is to impose $2n - 3$ conditions, and to require the curve of order $n - 2$, along which it cuts its asymptotes, to be orthic is to impose $\frac{1}{2}(n - 2)(n - 3)$ further conditions, in all $\frac{1}{2}n(n - 1)$. But $\frac{1}{2}n(n - 1)$ is the number of conditions required to make a curve of order n orthic.

Part Three—Pencils Determined by Two Orthic Curves and Orthocentric Sets of Points.

I. *Introduction.*

We shall now take up the study of the pencils of curves determined by two orthic curves. The main purpose of this investigation shall be to learn what we can about the figure of n^2 points in

Figure 2. The hypocycloid of class five and order six, which is enveloped by the asymptotes of curves in a pencil of orthic cubics.

which two orthic curves intersect. Such a figure of n^2 points we shall call an *Orthocentric Set*, or an *Orthocentric n^2-point.*

There is a well-known proposition that all the equilateral hyperbolas (orthic conics) which can be circumscribed to a given triangle pass through the orthocentre of the triangle. The four points, the vertices and the orthocentre of a triangle, or, what is the same thing, the intersections of two orthic curves of the second order, have the property that the line joining any two of them is perpendicular to the line joining the other two. The term orthocentric is applied to a set of four points related in this way. We wish to find out what metrical property distinguishes the n^2-point, in which two orthic curves of order n intersect.

II. *The Central Pencil and Its Orthocentric Set.*

The first generalization which we shall make is to show that any pair of points, α, β, together with their antipoints, α, $\bar{\beta}$ and β, $\bar{\alpha}$, form an orthocentric four-point. α and β determine a central pencil of orthic conics,

$$(x-\alpha)\,(x-\beta)=\tau\,(\bar{x}-\bar{\alpha})\,(\bar{x}-\beta),$$

and the antipoints are evidently on all the curves of the pencil.

If we consider τ as a parameter in the general equation of an orthic curve,

$$(x-a_1)\,(x-a_2)\ldots(x-a_n)=\tau\,(\bar{x}-\bar{a}_1)\,(\bar{x}-\bar{a}_2)\ldots(\bar{x}-\bar{a}_n),$$

we obtain the equation of all the curves of which $a_1 \ldots a_n$ is an *n-ad.* The points of the orthocentric n^2-point determined by this are the n real points a, and all their antipoints. But as the pencil is determined by the n real points it follows that:

Any n *points, with all their antipoints, form a central orthocentric n^2-point.*

The centroid of the n^2-point determined in this way is the centroid of the n real points. The real and imaginary foci of any curve are such a set of orthocentric points.

III. *The Pencil of Orthic Cubics through Five Points of a Circle. The Locus of Centres.*

We have seen that six points of a circle determine an orthic cubic curve. If the six points are t_1, t_2, t_3, t_4, t_5, t_6, then, as we have seen, the equation of the orthic cubic through them is

$$x^3-s_1x^2+s_2x-s_3+s_4\bar{x}-s_5\bar{x}^2+s_6\bar{x}^3=0.$$

If we replace t_6 by a variable parameter t, and put σ's for the elementary symmetrical combinations of $t_1 \ldots t_5$, we have

$$s_1=\sigma_1+t,\quad s_4=\sigma_4+t\sigma_3,$$

$$s_2=\sigma_2+t\sigma_1,\; s_6=\sigma_5+t\sigma_4,$$

$$s_3=\sigma_3+t\sigma_2,\; s_6=\sigma_5t.$$

If we make this substitution we get

$$x^3 - (\sigma_1 + t)x^2 + (\sigma_2 + t\sigma_1)x - (\sigma_3 + t\sigma_2)$$
$$+ (\sigma_4 + t\sigma_3)\bar{x} - (\sigma_5 + t\sigma_4)\bar{x}^2 + \sigma_6 t\bar{x}^3 = 0.$$

This is the equation of a pencil of orthic cubics through five points of a circle.

The centre of the curve through the six points is $x = \frac{1}{3}s_1$. If the sixth point move around the unit circle, this becomes

$$x = \tfrac{1}{3}(\sigma_1 + t).$$

This is the map equation of a circle. We have thus the theorem:

The locus of centres of the orthic cubics through five points of a circle is a circle. Its radius is one-third that of the given circle, and its centre is the point $\frac{1}{3}\sigma_1$.

M. Serret[1] gives an elegant synthetic proof of the theorem that the locus of centres of the curves of a pencil of equilaterals is a circle. I obtained the same result for orthic curves independently, and, as the analysis is so direct, it seems advisable to let it stand.

IV. *The Hypocycloid Enveloped by the Asymptotes.*

I shall now prove, for the pencil of orthic cubics through five points of a circle, a theorem which M. Serret[1] states without proof. The theorem referred to, when stated for orthic cubics of the pencil under discussion, becomes:

The curve enveloped by the asymptotes of all the orthic cubics through five points of a circle is an hypocycloid of order six and class five.

It is circumscribed to the centre circle of the pencil, and its cusps lie on a concentric circle five times as large.

We found that the equation of an asymptote, in terms of the six points where the curve cuts the unit circle, is

$$(x - \tfrac{1}{3}s_1) + t\sqrt{s_6}(\bar{x} - \tfrac{1}{3}s_5 s_6^{-1}) = 0.$$

If we replace t_6 by the parameter t, this becomes

$$x - \tfrac{1}{3}(\sigma_1 + t) + t\sqrt{\sigma_6 t}\,(\bar{x} - \tfrac{1}{3}\{\sigma_4\sigma_6^{-1} + t^{-1}\}) = 0.$$

[1] Sur les faisceaux reguliers et les équi'atères d'ordre *n*. Paul Serret, *Comptes Rendus*, 1895, t. 121, pp. 373-5.

we seek the curve enveloped by this line, as t runs around the unit circle.

For the sake of simplicity, let us refer this equation to a new system of coördinates, so chosen that the centre circle of the pencil becomes the new unit circle. The equation becomes

$$x - t + \sqrt[3]{\sigma_5 t}\,(\bar{x} - t^{-1}) = 0.$$

If now we take an axis of reals which makes $\sigma_5 = 1$, and also put τ^3 for t, we have

$$x\tau^{-1} - \tau^2 + \bar{x} - \tau^{-3} = 0.$$

The map equation of the curve enveloped by this line is obtained by equating to zero the result of differentiating with respect to τ. It is

$$x = 3\tau^{-2} - 2\tau^3.$$

This is a curve of double circular motion. The curve is of order six, for it meets any line,

$$x = \frac{a}{1 - \tau},$$

where

$$\frac{a}{1-\tau} = \frac{3}{\tau^2} - 2\tau^3,$$

or

$$2\tau^6 - 2\tau^5 - a\tau^2 + 3\tau - 3 = 0.$$

This gives six τ's, and, therefore, the curve is of the sixth order In order to determine the class of the curve, we must examine the equation of a tangent,

$$x\tau^{-1} - \tau^2 + \bar{x} - \tau^{-3} = 0.$$

This is of the fifth degree in the parameter, and there are, therefore, five tangents from any point x.

The stationary points, or cusps, are the points where the velocity of x is zero. For such a point we must have $D_\tau x = 0$, and at the same time $|\tau| = 1$. Both these conditions are satisfied by

$$\tau = \sqrt[5]{-1}.$$

The curve has, therefore, five real cusps; one when τ is each of the fifth roots of minus one.

If we put $\kappa^5 = -1$ we get a cusp.

$$x = 3\kappa^{-2} - 2\kappa^3,$$

$$\kappa^2 x = 5.$$

Since multiplication by κ^2 is equivalent to a rotation $\frac{4}{5}\pi$, we see that the locus of cusps is a circle, about the centre of the pencil, and five times as large as the centre circle. A rotation $\frac{2}{5}\pi$ sends each cusp into another, and so the cusps are equally spaced along the cusp circle. The intersections of the hypocycloid with the centre circle,

$$x\bar{x} = 1,$$

are obtained by solving $\bar{x} = x^{-1}$ for τ.
We have

$$x = 3\tau^{-2} - 2\tau^3,$$

and

$$\bar{x} = 3\tau^2 - 2\tau^{-3}.$$

The parameters of the points sought are the roots of

$$12\tau^5 - 6\tau^{10} - 6 = 0,$$

or of

$$(\tau^5 - 1)^2 = 0.$$

There are five pairs of coincident intersections. But since x cannot be less than 1, it follows that the curve is tangent to the circle in five places.

We have obtained this hypocycloid as the locus of one asymptote. But all three asymptotes envelop the same curve, for if we put ω for $\sqrt[5]{\sigma_5}$ we get

$$x = 3\omega\tau^{-2} - 2\tau^3.$$

This has a cusp at $\kappa^{12} x = 5$; it is, obviously, the same curve.

V. *Perpendicular Tangents of the Hypocycloid.*

The equation of a tangent to the curve is

$$x\tau^{-1} - \tau^2 + \bar{x} - \tau^{-3} = 0.$$

That of a perpendicular tangent,

$$-x\tau^{-1} - \tau^2 + \bar{x} + \tau^{-3} = 0.$$

These two lines meet at

$$x = \tau^{-2}.$$

In other words, *Perpendicular tangents to the envelope of the asymptotes meet on the centre circle.*

We have here a verification of the known property of the hypocycloid of this class, that the tangents from a point of the vertex circle are all real and form two regular pencils.[1]

VI. *The Orthocentric Nine-point of the Pencil through Five Points of a Circle, and the Extension to 2n—1 Points.*

Let us now consider the figure of nine orthocentric points, five of which are on a circle. The equation of the pencil of orthic cubics through five points of a circle is

$$x^3 - (\sigma_1 + t)x^2 + (\sigma_2 + t\sigma_1)x - (\sigma_3 + t\sigma_2)$$
$$+ (\sigma_4 + t\sigma_3)\bar{x} - (\sigma_5 + t\sigma_4)\bar{x}^2 + \sigma_5 t\bar{x}^3 = 0.$$

We know five of the points of the orthocentric nine-point determined by this pencil, and we seek the remaining four. Rewrite the above equation as

$$(x - t)\,(x^2 - \sigma_1 x + \sigma_2)$$
$$+ (t\bar{x} - 1)\,(\sigma_3 - \sigma_4\bar{x} + \sigma_5\bar{x}^2) = 0.$$

Now if both

$$x^2 - \sigma_1 x + \sigma_2$$

and

$$\bar{x}^2\sigma_5 - \sigma_4\bar{x} + \sigma_3$$

can become zero for conjugate values of x and $\bar{x}$, then those values are the coördinates of a real point which is on every curve of the pencil, and is one of the nine points. If we put $\sigma_5 = 1$, as we may, these two relations become

$$x^2 - \sigma_1 x + \sigma_2 = 0,$$

[1] F. Morley, "On the Epicycloid," *American Journal of Mathematics*, Vol. XIII, No. 2.

and

$$\bar{x}^2 - \overline{\sigma_1 x_1} + \bar{\sigma}_2 = 0.$$

These are conjugate equations and so can be satisfied by the coordinates of real points. Solving them, we get a pair of real points

$$x_1 = \frac{\sigma_1 + \sqrt{\sigma_1^2 - 4\sigma_2}}{2}, \quad \bar{x}_1 = \frac{\bar{\sigma}_1 + \sqrt{\bar{\sigma}_1^2 - 4\bar{\sigma}_2}}{2};$$

and

$$\frac{\sigma_1 - \sqrt{\sigma_1^2 - 4\sigma_2}}{2}, \quad \bar{x}_2 = \frac{\bar{\sigma}_1 - \sqrt{\bar{\sigma}_1^2 - 4\bar{\sigma}_2}}{2}.$$

But further, we notice that the antipoints, x_1, $\bar{x}_2$, and x_2, $\bar{x}_1$, of these make the equation of the pencil vanish for all values of the parameter. They are the remaining points of the orthocentric nine. This leads to the theorem that:

If five points of an orthocentric nine-point are on a circle, of the remaining four points two are real, two are imaginary; and these four form an orthocentric four-point.

The centroid of the nine points is

$$x = \tfrac{1}{9}\,(t_1 + \ .\ .\ t_5 + 2\sigma_1) = \tfrac{1}{3}\sigma_1.$$

This is the centre of the centre circle of the pencil.

We can extend these results to the case of n^2-points, $2n - 1$ of which lie on a circle.

The pencil of orthic curves of order n which go through $2n - 1$ points of the unit circle is given by

$$x^n - (\sigma_1 + t)\, x^{n-1} + (\sigma_2 + t\sigma_1)\, x^{n-2} - + \ .\ .\ .$$
$$(\sigma_{2n-2} + t\sigma_{2n-3})\, \bar{x}^{n-2} - (\sigma_{2n-1} + t\sigma_{2n-2})\, \bar{x}^{n-1} + \sigma_{2n} t \bar{x}^n = 0.$$

If we let $\sigma_{2n-1} = 1$, this becomes

$$(x - t)\, (x^{n-1} - \sigma_1 x^{n-2} + \sigma_2 x^{n-3} - + - \ .\ .\ \sigma_{n-1})$$
$$+ (\bar{x}t - 1)\, (\bar{x}^{n-1} - \bar{\sigma}_1 \bar{x}^{n-2} + - \ .\ .\ .\ \bar{\sigma}_{n-1}) = 0.$$

Now since the coefficients of $(x - t)$ and $(\bar{x}t - 1)$ are conjugate forms, there are $n - 1$ real points, in addition to the points t_1, t_2 $.\ .\ .\ .\ t_{2n-1}$, which are on all the curves of the pencil. Further, all the antipoints obtained by pairing these in all possible ways satisfy the equation for all values of t. Now we know that the $(n-1)^2$ points thus found form an orthocentric set. We are now in a position to state the following general theorem:

If $2n-1$ points of an orthocentric set of n^2-points lie on a circle, then the remaining $(n-1)^2$ points of the figure form a central orthocentric set of which $n-1$ points are real.

The vectors of the $n-1$ real points are the roots of

$$x^{n-1} - \sigma_1 x^{n-2} + \sigma_2 x^{n-3} - + \ldots \sigma_{n-1} = 0.$$

VII. *The Pencil Determined by Any Two Orthic Curves.*

We are now ready to consider the most general pencil of orthic curves. Form the equation

$$x^n - (a_1 + ta'_1)\, x^{n-1} + (a_2 + ta'_2)\, x^{n-2} - + \ldots$$
$$\dot{-} (a_{2n-1} + ta'_{2n-1})\, \bar{x}^{n-1} + t\bar{x}^n = 0,$$

where t is a parameter which has the absolute value unity. Now for every value of t this represents a real orthic curve of the n^{th} order, provided

$$\overline{a_v + ta'_v} = (a_{2n-v} + ta'_{2n-v})t^{-1},$$

or

$$|\, \bar{a}_v - a'_{2n-v} \,| = |\, \bar{a}'_v - a_{2n-v} \,|\,.$$

For if this holds, the equation can be put in the known form

$$(x - a_1)\,(x - a_2) \ldots = \tau_1\, (\bar{x} - \bar{a}_1)\,(\bar{x} - \bar{a}_2) \ldots (\bar{x} - \bar{a}_n).$$

Now let

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} - + \ldots - a_{2n-1}\bar{x}^{n-1} + a_{2n}\bar{x}^n = 0,$$

and

$$x^n - a'_1 x^{n-1} + a'_2 x^{n-2} - + \ldots - a'_{2n-1}\bar{x}^{n-1} + a'_{2n}\bar{x}^n = 0,$$

be the equations of any two real orthic curves. Then

$$a_{2n} = t_1,\ a'_{2n} = t_2,$$

and

$$\bar{a}_v = a_{2n-v} t_1^{-1},\ \bar{a}'_v = a'_{2n-v} t_2^{-1}.$$

We can choose the a's in such a way that the pencil will include the given curves, (1) and (2), for the $4n - 2$ equations

$$a_v = a_v + t_1 a'_v,$$
$$a'_v = a_v + t_2 a'_v, \qquad v = 1 \ . \ . \ . \ 2n-1$$

just suffice. We must show now that when the coefficients are determined as above, all the curves of the pencil are real.

Now we have

$$a_v = a_v + t_1 a'_v,$$

and

$$a_{2n-v} = a_{2n-v} + t_1 a'_{2n-v}.$$

From these, we get

$$\overline{a_v + t_1 a'_v} = \bar{a}_v = a_{2n-v} t_1^{-1} = a_{2n-v} t_1^{-1} + a'_{2n-v},$$

and therefore

$$\bar{a}_v - a'_{2n-v} = (a_{2n-v} - \bar{a}'_v) t_1^{-1}.$$

But this is the condition that every curve of the pencil be real. It is clear that no curve not orthic can be included in the pencil. So we see that:

Any two real orthic curves of order n *determine a pencil of real curves of the same order, all of which are orthic.*

VIII. *The Serret Circle, or Locus of Centres.*

M. Serret's theorem (Part Three, IV) on the locus of centres is easily verified. The centre of any curve of the pencil is

$$x = \tfrac{1}{n}(a_1 + t a'_1).$$

Now if t is regarded as a parameter, this is the map equation of a circle with its centre at

$$x = \tfrac{1}{n} a_1.$$

The locus of centres of the most general pencil of orthic curves is a circle.

In the special case where n of the intersections of the pencil are at infinity, the locus of centres degenerates into a right line. A pencil of this type may be written

$$x^n - \frac{a_1 - \lambda a'_1}{1 - \lambda} x^{n-1} + \ . \ . \ . \ \frac{a_{2n-1} - \lambda a'_{2n-1}}{1 - \lambda} \bar{x}^{n-1} + \bar{x}^n = 0,$$

where λ is a real parameter. The locus of centres is

$$x = \frac{1}{n}\left(\frac{a_1 - \lambda a'_1}{1 - \lambda}\right).$$

The elimination of λ from this and its conjugate gives

$$x\,(a_{2n-1} - a'_{2n-1}) - \bar{x}\,(a_1 - a'_1) + \tfrac{1}{n}\,(a_1 a'_{2n-1} - a'_1 a_{2n-1}) = 0,$$

the equation of a right line.

IX. *The Hypocycloid Enveloped by the Asymptotes.*

Let us now seek the curve enveloped by the asymptotes of the curves of a general pencil. The equation of an asymptote of the curve given by t_1 is

$$x - \tfrac{1}{n}\,(a_1 + t_1 a'_1) + \sqrt[n]{t_1}\left\{\,\bar{x} - \tfrac{1}{n}\,(a_{2n-1} t^{-1} + a'_{2n-1})\,\right\} = 0,$$

or

$$x - \tfrac{1}{n}\,(a_1 + t_1 a'_1) + \sqrt[n]{t_1}\left\{\,\bar{x} - \overline{(a_1 + t_1 a'_1)}\,\right\} = 0.$$

For convenience, transform to the centre of the pencil, $\frac{1}{n}a$, as a new origin. The equation becomes

$$x - \tfrac{1}{n}\,a'_1 t_1 + \sqrt[n]{t_1}(x - \tfrac{1}{n}\overline{a'_1 t_1}) = 0.$$

Putting $\tau^n = t$, we get

$$x - \tfrac{1}{n}a'_1\tau^n + \tau\bar{x} - \tfrac{1}{n}\bar{a}'_1\tau^{1-n} = 0,$$

and finally,

$$x\tau^{-1} - \tfrac{1}{n}a'_1\tau^{n-1} + \bar{x} - \tfrac{1}{n}\bar{a}'_1\tau^{-n} = 0.$$

Now the map equation of the curve enveloped by this line as τ varies is

$$nx = n\bar{a}'_1\tau^{1-n} + (1 - n)\,a'_1\tau^n.$$

Now this equation represents a curve of double circular motion. We know that

$$\bar{a}'_1 = a'_1 t_2^{-1},$$

and using it we get

$$nx = na'_1 t_2^{-1}\tau^{1-n} + (1 - n)\,a'_1\tau^n.$$

Now if we make t_2 real, and then regard the centre circle as the unit circle, *i.e.*, adopt $\left| \frac{a'_1}{n} \right|$ as the unit length, the equation takes the form

$$x = n\tau^{1-n} + (1 - n)\tau^n.$$

This is the equation of an hypocycloid of the kind found as the locus of asymptotes of a special pencil of orthic cubics. Its vertex circle is the centre circle of the pencil. It has cusps when

$$D_\tau x = 0,$$

and

$$|\tau| = 1,$$

simultaneously, or when

$$\tau^{2n-1} + 1 = 0.$$

The parameters of the cusps are the $2n - 1^{st}$ roots of -1.
If we let $\kappa^{2n-1} = -1$, a cusp is

$$x = n\kappa^{1-n} + (1 - n)\,\kappa^n$$

or

$$x\kappa^{n\ 1} = n - (1 - n)$$

The absolute value of a cusp is, therefore, $2n - 1$.

Since the equation of a tangent,

$$x - \tfrac{1}{n}a'_1\tau^n + \tau\bar{x} - \tfrac{1}{n}\tau^{1-n}\,\bar{a}'_1 = 0$$

is of the $2n - 1^{st}$ degree in the parameter τ, the hypocycloid is of class $2n - 1$. If we eliminate x between the equation of the curve and the equation of any line,

$$x = \frac{a}{1 - \tau}$$

we get an equation of the $2n^{th}$ degree to determine the parameters of the points of intersection. The curve meets any line in $2n$ points, and is therefore of order $2n$. We have now established analytically the theorem stated by M. Serret, as far as orthic curves are concerned. It is:

The curve enveloped by the asymptotes of a pencil of orthic curves of order n *is an hypocycloid of order* 2n, *and of class* 2n — 1. *Its vertex circle is the centre circle of the pencil, and its cusp circle is concentric with that circle, and* 2n — 1 *times as large.*

If we bear in mind that any difference between an orthic curve and any equilateral does not affect the terms of the n^{th} and $n - 1^{st}$ degrees of the equation, we see that the method of proof used above is applicable to equilaterals in general.

X. *A Circle Determined by Any Odd Number of Points.*

It is a well-known proposition that the centres of the equilateral hyperbolas circumscribed to a triangle lie on the circle through the mid-points of the sides of the triangle. This circle is usually called the Feuerbach, or nine-point circle of the triangle. Now we have seen that an orthic curve of order n may be made to satisfy $2n$ linear conditions; it follows that any odd number, $2n - 1$, of points determine a pencil of orthic curves of the n^{th} order. Connected with this pencil is the centre circle, or, as I propose to call it, the Serret circle, which is, in a sense, the generalized nine-point circle.

Every figure of an odd number of points has connected with it a unique circle, the Serret circle, which in the case of three points is identical with the nine-point circle of Feuerbach.

Further, every odd number of points, $2n - 1$, determine the pencil of orthic curves through them, and therefore the remaining $(n - 1)^2$ points of the orthocentric n^2-point. In the case of three given points, this set of $(n - 1)^2$ points is a single point, the orthocentre of the given points. So we are led to the theorem:

To every figure of $2n - 1$ *points belongs a figure of* $(n - 1)^2$ *points.*

In one sense the Serret circle belongs to n^2 points, but of these only $2n - 1$ may be taken at random.

XI. *A Point Determined by An Even Number of Points.*

Now consider an even number, $2n$, of points which do not belong to an orthocentric n^2-point. There is a pencil of orthic curves through every $2n - 1$ points which can be selected from them, or $2n$ pencils in all. Now these pencils give rise to $2n - 1$ Serret circles, but there is one orthic curve through all $2n$ points and its centre is on each of the circles. We have, therefore, the result:

The $2n$ *Serret circles, given by all the sets of* $2n - 1$ *among* $2n$ *points, meet in a point.*

XII. *The Relation of the Orthocentric* n^2*-point to the Circle of Centres.*

In section VIII we obtained the pencil of orthic curves determined by the two given curves,

$$(1)\ x^n - a_1x^{n-1} + a_2x^{n-2} - + \ldots - a_{2n-1}\bar{x}^{n-1} + a_{2n}\bar{x}^n = 0,$$

and

$$(2)\ x^n - a'_1x^{n-1} + a'_2x^{n-2} - + \ldots - a'_{2n-1}\bar{x}^{n-1} + a'_{2n}\bar{x}^n = 0.$$

We now wish to show that the centroid of the orthocentric n^2-point in which these two curves intersect is the centre of the centre circle of the pencil. If we rewrite (1) and (2) in terms of $\bar{x}$ we get

$$(1) \qquad (\bar{x} - \bar{x}_1)(\bar{x} - \bar{x}_2) \ldots (\bar{x} - \bar{x}_n) = 0,$$

and

$$(2) \qquad (\bar{x} - \bar{x}'_1)(\bar{x} - \bar{x}'_2) \ldots (\bar{x} - \bar{x}'_n) = 0.$$

If the s's refer to the elementary symmetrical functions of the roots, we have

$$s_i = a_{2n-i},\ s'_i = a'_{2n-i},\ i = 1, 2, \ldots n - 1.$$

$$s_n = -(x^n - a_1x^{n-1} + a_2x^{n-2} \ldots a_n)\, a_{2n}^{-1},$$

$$s'_n = -(x^n - a'_1x^{n-1} + a'_2x^{n-2} \ldots a'_n)\, a'_{2n}{}^{-1}.$$

Now the eliminant of x between these two equations is

$$\begin{array}{c}
(\bar{x}_1 - \bar{x}'_1)(\bar{x}_1 - \bar{x}'_2) \ldots (\bar{x}_1 - \bar{x}'_n).\\
(\bar{x}_2 - \bar{x}'_1)(\bar{x}_2 - \bar{x}'_2) \ldots (\bar{x}_2 - \bar{x}'_n).\\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
(\bar{x}_n - \bar{x}'_1)(\bar{x}_n - \bar{x}'_2) \ldots (\bar{x}_n - \bar{x}'_n) = 0.
\end{array}$$

This is a function of degree n^2 in x, and as x occurs in s_n and s'_n alone, we need consider those terms alone in which the products s_n and s'_n appear. These are:

$$s_n^n - ns_n^{n-1}s'_n + \frac{n.n-1}{2} s_n^{n-2}s'_n{}^2 \ldots s'_n{}^n,$$

or

$$(s_n - s'_n)^n;$$

or, in terms of x,

$$\left\{ x^n \left(\frac{a_{2n} - a'_{2n}}{a_{2n}a'_{2n}}\right) - x^{n-1} \left(\frac{a_{2n}a'_1 - a'_{2n}a_1}{a_{2n}a'_{2n}}\right) + \ldots \left(\frac{a_{2n}a'_n - a'_{2n}a_n}{a_{2n}a'_{2n}}\right) \right\}^n.$$

When this is expanded and arranged in powers of x the first and second terms are

$$x^{n^2}\left(\frac{a_{2n} - a'_{2n}}{a_{2n}a'_{2n}}\right)^n - n\left(\frac{a_{2n} - a'_{2n}}{a_{2n}a'_{2n}}\right)^{n-1}\left(\frac{a_{2n}a'_1 - a'_{2n}a_1}{a_{2n}a'_{2n}}\right)x^{n^2-1}.$$

Now the sum of the roots is

$$\sigma_1 = n\,\frac{a_{2n}a'_1 - a'_{2n}a_1}{a_{2n} - a'_{2n}},$$

and their centroid is

$$\frac{1}{n^2}\,\sigma_1 = \frac{1}{n}\,\frac{a_{2n}a'_1 - a'_{2n}a_1}{a_{2n} - a'_{2n}} = x'.$$

Now $a_{2n} = t_1$, and $a'_{2n} = t_2$, and we have also the relations

$$a_v = a_v + t_1 a'_v,$$

and

$$a'_v = a_v + t_2 a'_v,$$

from which we obtain

$$x' = \tfrac{1}{n}\,\frac{t_1a_1 + t_1t_2a'_1 - t_2a_1 - t_1t_2a'_1}{t_1 - t_2}$$

$$= \tfrac{1}{n}a_1.$$

But this is precisely the centre of the centre circle

$$x = \tfrac{1}{n}\,(a_1 + ta'_1).$$

We are thus enabled to conclude with the general theorem:

The centroid of an orthocentric set of points is the centre of the centre circle of the pencil of orthic curves through those points.

Johns Hopkins University, May 20, 1904.